COMMUTATIVE ALGEBRAS OF TOEPLITZ OPERATORS AND LAGRANGIAN FOLIATIONS

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To Nikolai Vasilevski on the occasion of his 60th birthday

ABSTRACT. Let D be a homogeneous bounded domain of $\mathbb{C}^n$ and $\mathcal{A}$ a set of (anti–Wick) symbols that defines a commutative algebra of Toeplitz operators on every weighted Bergman space of D. We prove that if $\mathcal{A}$ is rich enough, then it has an underlying geometric structure given by a Lagrangian foliation.

1. INTRODUCTION

In recent work, Vasilevski and his collaborators discovered unexpected commutative algebras of Toeplitz operators acting on weighted Bergman spaces on the unit disk (see [7] and [8]). It was even possibly to classify all such algebras as long as they are assumed commutative for every weight and suitable richness conditions hold; the latter ensure that the (anti–Wick) symbols that define the Toeplitz operators have enough infinitesimal linear independence. We refer to [3] for further details.

Latter on, it was found out that the phenomenon of the existence of nontrivial commutative algebras of Toeplitz operators extends to the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. There exists at least $n+2$ nonequivalent such commutative algebras which were exhibited explicitly in [5] and [6]. The discovery of such commutative algebras of Toeplitz operators on $\mathbb{B}^n$ was closely related to a profound understanding of the geometry of this domain. The description of these commutative algebras of Toeplitz operators involved Lagrangian foliations, i.e. by Lagrangian submanifolds (see Section 2 for detailed definitions), with distinguished geometric properties.

In order to completely understand the commutative algebras of Toeplitz operators on $\mathbb{B}^n$ and other domains, it is necessary to determine the general features that produce such algebras. In particular, there is the question as to whether or not the commutative algebras of Toeplitz operators have always a geometric origin. More precisely, whether or not they are always given by a Lagrangian foliation.

The main goal of this paper is to prove that, on a homogeneous domain, commutative algebras of Toeplitz operators are always obtained from Lagrangian foliations. This is so at least when the commutativity holds on every weighted Bergman space and a suitable richness condition on the symbols holds. To state this claim we use the following notation. We denote by $C^\infty(M)$ the space of smooth complex-valued functions on a manifold M and by $C_b^\infty(M)$ the subspace of bounded functions. For a given foliation $\mathcal{F}$ of a manifold M, we will denote by $\mathcal{A}_{\mathcal{F}}(M)$ the vector subspace

of $C^\infty(M)$ that consists of those functions which are constant along every leaf of $\mathcal{F}$. Our main result is the following.

Main Theorem . *Let D be bounded domain in $\mathbb{C}^n$ and $\mathcal{A}$ a vector subspace of $C_b^\infty(D)$ which is closed under complex conjugation. If for every $h \in (0,1)$ the Toeplitz operator algebra $\mathcal{T}_h(\mathcal{A})$, acting on the weighted Bergman space $\mathcal{A}_h^2(D)$, is commutative and $\mathcal{A}$ satisfies the following richness condition:*

- *for some closed nowhere dense subset $S \subseteq D$ and for every $p \in D \setminus S$ there exist real-valued elements $a_1, \ldots, a_n \in \mathcal{A}$ such that $da_{1p}, \ldots, da_{np}$ are linearly independent over $\mathbb{R}$,*

then, there is a Lagrangian foliation $\mathcal{F}$ of $D \setminus S$ such that $\mathcal{A}|_{D \setminus S} \subseteq \mathcal{A}_\mathcal{F}(D \setminus S)$. In other words, every element of $\mathcal{A}$ is constant along the leaves of $\mathcal{F}$.

To obtain this result we prove the following characterization of spaces of functions which define commutative algebras with respect to the Poisson brackets in a symplectic manifold.

Theorem 1.1. *Let M be a $2n$-dimensional symplectic manifold and $\mathcal{A}$ a vector subspace of $C^\infty(M)$ which is closed under complex conjugation. Suppose that the following conditions are satisfied:*

1. *$\mathcal{A}$ is a commutative algebra for the Poisson brackets of M, i.e. $\{a, b\} = 0$ for every $a, b \in \mathcal{A}$, and*
2. *for every $p \in M$ there exist real-valued elements $a_1, \ldots, a_n \in \mathcal{A}$ such that $da_{1p}, \ldots, da_{np}$ are linearly independent over $\mathbb{R}$.*

Then, there is a Lagrangian foliation $\mathcal{F}$ of M such that $\mathcal{A} \subseteq \mathcal{A}_\mathcal{F}(M)$.

This together with Berezin's correspondence principle (see Section 3) allows us to prove the Main Theorem.

We note that it is a known fact that for a Lagrangian foliation $\mathcal{F}$ in a symplectic manifold M, the vector subspace $\mathcal{A}_\mathcal{F}(M)$ is a commutative algebra with respect to the Poisson brackets. More precisely we have the following result.

Proposition 1.2. *If $\mathcal{F}$ is a Lagrangian foliation of a symplectic manifold M, then:*
$$\{a, b\} = 0,$$
for every $a, b \in \mathcal{A}_\mathcal{F}(M)$.

Hence, Theorem 1.1 can also be thought of as a converse of Proposition 1.2.

2. Preliminaries on symplectic geometry and foliations

The goal of this section is to establish some notation and state some very well known results on symplectic geometry and foliations.

For the next remarks on symplectic geometry we refer to [4] for further details.

Let M be a symplectic manifold with symplectic form ω. Being nondegenerate, the symplectic form ω defines an isomorphism between the tangent and cotangent spaces. More precisely, we have the following elementary fact.

Lemma 2.1. *For every $p \in M$ the map:*
$$\begin{aligned} T_pM &\rightarrow T_p^*M \\ v &\rightarrow \omega(v, \cdot), \end{aligned}$$
is an isomorphism of vector spaces.

This remark allows us to construct vector fields associated to 1-forms. In particular, for a complex-valued smooth function f defined over M, we define the Hamiltonian field associated to f as the smooth vector field over M that satisfies the identity:

$$df(X) = \omega(X_f, X),$$

for every vector field X over M. The Poisson brackets of two complex-valued smooth functions f, g over M is then given as the smooth function:

$$\{f, g\} = \omega(X_f, X_g) = df(X_g).$$

The following well known result relates the Poisson brackets on smooth functions to the Lie brackets of vector fields.

Lemma 2.2. *The Poisson brackets define a Lie algebra structure on the space $C^\infty(M)$. Also, we have the identity:*

$$[X_f, X_g] = X_{\{f,g\}},$$

for every $f, g \in C^\infty(M)$. In particular, the assignment:

$$f \mapsto X_f$$

is an isomorphism of Lie algebras onto the Lie algebra of Hamiltonian fields.

Another important object in our discussion is given by the notion of a foliation $\mathcal{F}$ of a manifold M. This is given as a decomposition into connected submanifolds which is locally given by submersions. More precisely, we define a foliated chart for M as a smooth submersion $\varphi : U \subseteq M \to \mathbb{R}^k$ from an open subset of M onto an open subset of $\mathbb{R}^k$. Given two such foliated charts φ, ψ, defined on open subsets U, V respectively, we will say that they are compatible if there is a smooth diffeomorphism $\xi : \varphi(U \cap V) \to \psi(U \cap V)$ such that $\xi \circ \varphi = \psi$ on $U \cap V$. A foliated atlas for M is a family of compatible foliated charts whose domains cover M; note that we also need k above to be the same for all the foliated charts. For any such foliated atlas, we define the plaques as the connected components of the fibers of its foliated charts. With these plaques we define the following equivalence relation in M:

$$x \sim y \iff \quad \text{there is a sequence of plaques } (P_j)_{j=0}^l$$
$$\text{of the foliated atlas, such that } x \in P_0, \ y \in P_l,$$
$$\text{and } P_{j-1} \cap P_j \neq \phi \text{ for every } j = 1, \ldots, l$$

The equivalence classes of such an equivalence relation are called the leaves of the foliation, which are easily seen to be submanifolds of M. For further details on this definition and some of its consequences and properties we refer to [2].

Finally, a foliation $\mathcal{F}$ in a symplectic manifold M is called Lagrangian if its leaves are Lagrangian submanifolds of M.

3. Berezin's correspondence principle for bounded domains

In the rest of this section D denotes a homogeneous bounded domain of $\mathbb{C}^n$. We now recollect some facts on the analysis of D leading to Berezin's correspondence principle; we refer to [1] for further details.

For every $h \in (0, 1)$, let us denote by $\mathcal{A}_h^2(D)$ the weighted Bergman space defined as the closed subspace of holomorphic functions in $L^2(D, d\mu_h)$. Here, $d\mu_h$ denotes the weighted volume element obtained from the Bergman kernel. If we denote

by $B_D^{(h)} : L^2(D, d\mu_h) \to \mathcal{A}_h^2(D)$ the orthogonal projection, then for every $a \in L^\infty(D, d\mu_h)$ the Toeplitz operator $T_a^{(h)}$ with (anti–Wick) symbol a is given by the assignment:

$$T_a^{(h)} : \mathcal{A}_h^2(D) \to \mathcal{A}_h^2(D)$$
$$\varphi \mapsto B_D^{(h)}(a\varphi).$$

For any such (anti–Wick) symbol a and its associated Toeplitz operator $T_a^{(h)}$, Berezin [1] constructed a (Wick) symbol $\widetilde{a}_h : D \times \overline{D} \to \mathbb{C}$ defined so that the relation:

$$T_a^{(h)}(\varphi)(z) = \int_D \widetilde{a}_h(z, \overline{\zeta})\varphi(\zeta)F_h(\zeta, \overline{\zeta})d\mu(\zeta),$$

holds for every $\varphi \in \mathcal{A}_h^2(D)$, where $d\mu$ is the (weightless) Bergman volume and F_h is a suitable kernel defined in terms of the Bergman kernel and depending on h. This provides the means to describe the algebra of Toeplitz operators as a suitable algebra of functions. To achieve this, one defines a $*$-product of two Wick symbols $\widetilde{a}_h, \widetilde{b}_h$ as the symbol given by:

$$(\widetilde{a}_h * \widetilde{b}_h)(z, \overline{z}) = \int_D \widetilde{a}_h(z, \overline{\zeta})\widetilde{b}_h(\zeta, \overline{z})G_h(\zeta, \overline{\zeta}, z, \overline{z})d\mu(\zeta),$$

where G_h is again some kernel defined in terms of the Bergman kernel and depending on h. We denote by $\widetilde{\mathcal{A}}_h$ the vector space of Wick symbols associated to anti–Wick symbols that belong to $C_b^\infty(D)$. Then $\widetilde{\mathcal{A}}_h$ can be considered as an algebra for the $*$-product defined above.

Berezin's correspondence principle is then stated as follows.

Theorem 3.1 (Berezin [1])**.** *Let D be a homogeneous bounded domain of $\mathbb{C}^n$. Then, the map given by:*

$$\mathcal{T}_h(C_b^\infty(D)) \to \widetilde{\mathcal{A}}_h$$
$$T_a^{(h)} \mapsto \widetilde{a}_h$$

is an isomorphism of algebras, where $\mathcal{T}_h(C_b^\infty(D))$ is the Toeplitz operator algebra defined by bounded smooth symbols. Furthermore, the following correspondence principle is satisfied:

$$(\widetilde{a}_h * \widetilde{b}_h - \widetilde{b}_h * \widetilde{a}_h)(z, \overline{z}) = ih\{a, b\}(z) + O(h^2),$$

for every $h \in (0, 1)$ and every $a, b \in C_b^\infty(D)$.

4. Proofs of the main results

For the sake of completeness, we present here the proof of Proposition 1.2.

Proof of Proposition 1.2. For a given $a \in \mathcal{A}_\mathcal{F}(M)$, the condition of a being constant along the leaves of $\mathcal{F}$ implies that:

$$\omega(X_a, X) = da(X) = 0,$$

for every smooth vector field X tangent to $\mathcal{F}$. Since the foliation $\mathcal{F}$ is Lagrangian, at every $p \in M$ the space $T_p\mathcal{F}$ is a maximal isotropic subspace for ω and so the above identity shows that $(X_a)_p$ belongs to $T_p\mathcal{F}$. Hence, for every $a \in \mathcal{A}_\mathcal{F}(M)$ the vector field X_a is tangent to the foliation $\mathcal{F}$. From this we conclude that:

$$\{a, b\} = da(X_b) = 0,$$

for every $a, b \in \mathcal{A}_\mathcal{F}(M)$. $\square$

We now prove our result on commutative Poisson algebras and Lagrangian foliations.

Proof of Theorem 1.1. Let us consider a subspace $\mathcal{A}$ of $C^\infty(M)$ as in the hypotheses of Theorem 1.1. For every $p \in M$ define the vector subspace of $T_p M$ given by:

$$E_p = \{(X_a)_p : a \in \mathcal{A}\}.$$

We will now prove that $E = \cup_{p \in M} E_p$ is a smooth n-distribution over M; in other words, that in a neighborhood of every point the fibers of E are spanned by n smooth vector fields which are pointwise linearly indepent in such neighborhood.

First note that since the assignment $a \mapsto X_a$ is linear, every set E_p is a subspace of $T_p M$. Furthermore, being $\mathcal{A}$ commutative for the Poisson brackets, it follows that:

$$\omega(X_a, X_b) = \{a, b\} = 0,$$

for every $a, b \in \mathcal{A}$. In particular, E_p is an isotropic subspace for ω and so has dimension at most n.

On the other hand, for every $p \in M$ we can choose smooth functions $a_1, \ldots, a_n \in \mathcal{A}$ whose differentials are linearly independent at p. Hence, it follows from Lemma 2.1 that the elements $(X_{a_1})_p, \ldots, (X_{a_n})_p$ are also linearly independent at p, thus showing that E_p has dimension exactly n. By continuity, the chosen vector fields $X_{a_1}, \ldots, X_{a_n}$ are linearly independent in a neighborhood of p and so their values span E_q for every q in such neighborhood. Hence, E is indeed an n-distribution and our proof shows that its fibers are Lagrangian.

By Lemma 2.2, the assignment $a \mapsto X_a$ is a homomorphism of Lie algebras, and so the commutativity of $\mathcal{A}$ with respect to the Poisson brackets implies that the vector fields X_a commute with each other for $a \in \mathcal{A}$. Since the latter span the distribution E we conclude that E is involutive and, by Frobenius' Theorem (see [9]), it is integrable to some foliation $\mathcal{F}$. Note that $\mathcal{F}$ is necessarily Lagrangian.

It is enough to prove that every $a \in \mathcal{A}$ is constant along the leaves of $\mathcal{F}$. But by hypothesis we have:

$$da(X_b) = \{a, b\} = 0,$$

for every $a, b \in \mathcal{A}$. Since the vector fields X_b ($b \in \mathcal{A}$) define the elements of E at the fiber level we conclude that for any given $a \in \mathcal{A}$ we have:

$$da(X) = 0$$

for every vector field X tangent to $\mathcal{F}$. This implies that every $a \in \mathcal{A}$ is constant along the leaves of $\mathcal{F}$. $\square$

Finally, we establish the necessity of having a Lagrangian foliation underlying to every commutative algebra of Toeplitz operators with sufficiently rich (anti–Wick) symbols.

Proof of the Main Theorem. For every $h \in (0, 1)$, let us denote by $\widetilde{\mathcal{A}}_h(\mathcal{A})$ the algebra of Wick symbols corresponding to anti–Wick symbols $a \in \mathcal{A}$. By the first part of Theorem 3.1, the algebra $\widetilde{\mathcal{A}}_h(\mathcal{A})$ is commutative. Hence, by the correspondence principle stated in the second part of Theorem 3.1 it follows that $\mathcal{A}$ is commutative with respect to the Poisson brackets $\{\cdot, \cdot\}$. The result now follows from Theorem 1.1. $\square$

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